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A METHOD OF ANALYSIS FOR LENS AND MIRROR SYSTEMS

by

R. J. Davis[1], S. E. Strom[2], and K. M. Strom[3] ~~Jan. 30, 1961~~ In its

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The Smithsonian Astrophysical Observatory is participating in the Orbiting Astronomical Observatories program of the National Aeronautics and Space Administration to develop an optical-electronic system that will combine wide-band television with photometry and slitless spectroscopy in making a survey of the sky between 1000 and 3000 angstroms.

The system is basically a television telescope. The detector will be a modification of a Westinghouse ebicon tube for use in the ultraviolet. A photoemissive substance is deposited on the inside of the faceplate, which must be a lens transparent to the ultraviolet wavelength being studied. Photoelectrons from this surface are accelerated through 15 or 20 kilovolts and imaged upon the target electrode, in which the electron bombardment induces conductivity. The electroconductive image can then be read off the target by the usual vidicon technique. The resolution, which is determined by the limitations of beam-scanning techniques, is at present limited to about 100 microns throughout an area of 2.5 centimeters square on the photocathode.

The design of such a telescope involves the evaluation of various optical designs. It must be mechanically simple if it is to operate properly after being subjected to the rocket and space environment. It must consist entirely of reflection optics, except for the television faceplate itself and possibly a small optical filter in front of it.

We have developed a ray-tracing program for the IBM-704 electronic computer to evaluate various systems that have been proposed to meet our requirements. Because these techniques have more general applicability, we are reporting them here for the benefit of others who may wish to use them.

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General Method

The general vector equation for refraction and reflection at an optical surface is:

$$\underline{I} \times \underline{N} = n(\underline{R} \times \underline{N}) . \tag{1}$$

The following conventions hold:

$\underline{I}$ is the unit vector along the incident ray; $\underline{I} = A\underline{i} + B\underline{j} + C\underline{k}$, where A, B, and C are the direction cosines of the incident ray. This vector must be normalized, i.e., $A^2 + B^2 + C^2 = 1$.

$\underline{N}$ is the vector along the normal to the optical surface under consideration; this ray need not be normalized since it appears on both sides of the equation, $\underline{N} = P_x\underline{i} + P_y\underline{j} + P_z\underline{k}$, where P_x, P_y, and P_z are the partial derivatives of the function representing the optical surface $[P(x, y, z) = 0]$.

n is the ratio of the index of refraction of the second medium to that of the first; for reflection, $n = -1$.

$\underline{R}$ is a unit vector along the reflected (or refracted) ray; $\underline{R} = R\underline{i} + S\underline{j} + T\underline{k}$, where R, S, and T are the direction cosines of the ray.

The equation to be solved is then:

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A & B & C \\ P_x & P_y & P_z \end{vmatrix} = n \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ R & S & T \\ P_x & P_y & P_z \end{vmatrix} . \tag{2}$$

Or, when we equate the components, the equation takes the form:

$$\left.\begin{aligned} BP_z - CP_y &= n(SP_z - TP_y) \\ AP_z - CP_x &= n(RP_z - TP_x) \\ AP_y - BP_x &= n(RP_y - SP_x) \end{aligned}\right\} \tag{3}$$

Since equations (3) are not linearly independent, the other condition necessary both physically and mathematically to obtain a unique solution is:

$$R^2 + S^2 + T^2 = 1 . \tag{4}$$

$$\frac{x - x_o}{A} = \frac{y - y_o}{B} = \frac{z - z_o}{C}, \tag{8}$$

where (x_o, y_o, z_o) is any point on the incident ray.

We then evaluate the partial derivatives of the surface at the points of intersection, and carry out the solution for the values of R, S, and T that correspond to each set of A, B, and C. These values of R, S, and T can then be used as the direction cosines of the rays incident upon a second surface. Thus, the R, S, and T that have been calculated become analogous to the original A, B, and C. The rays are traced through the system in this manner until their intersections with the final optical focal surface have been calculated. Knowing these intersections, we can calculate both the size and the shape of the image if the incident rays are appropriately chosen and if their number is great enough.

All signs of the square roots involved in this solution are determined by the physical solution desired.

The Case of the Off-axis Paraboloid

One of the optical systems we have considered in our design of telescopes to be used in the Orbiting Astronomical Observatories program is the off-axis paraboloid. Such an instrument is illustrated in Figure 3.

The equation of the surface to be considered here is:

$$x^2 + y^2 = 4fz, \tag{9}$$

where f is the focal length of the paraboloid. The incident rays are given by:

$$\underset{\sim}{I} = (\sin\gamma \cos a)\,\underset{\sim}{i} + (\sin\gamma \sin a)\,\underset{\sim}{j} + (\cos\gamma)\,\underset{\sim}{k}, \tag{10}$$

where γ is the angle the ray makes with the z-axis, and a is the angle the ray makes with the x-axis, both angles being measured according to the conventions in Figure 4. These components satisfy the normalization condition. The equation of the normal is:

$$\underset{\sim}{N} = 2x\underset{\sim}{i} + 2y\underset{\sim}{j} - 4f\underset{\sim}{k}. \tag{11}$$

The optical system is set up in a rectangular coordinate system (x, y, z), with the first surface placed at the origin. Further surfaces are first written in convenient rectangular coordinate systems (x'', y'', z''). This is shown in Figure 1. In order to obtain a solution, we use the following transformation equations:

$$\left.\begin{aligned} x' &= x - x_1 \\ y' &= y - y_1 \\ z' &= z - z_1 \end{aligned}\right\} \qquad (5)$$

for translation, where (x_1, y_1, z_1) is the position of the surface with respect to the origin of the coordinate system (x, y, z); the matrix equation:

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \begin{bmatrix} \cos \alpha_1 & \cos \beta_1 & \cos \gamma_1 \\ \cos \alpha_2 & \cos \beta_2 & \cos \gamma_2 \\ \cos \alpha_3 & \cos \beta_3 & \cos \gamma_3 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \qquad (6)$$

is used for rotation. The angle convention is given in equations (7):

$$\left.\begin{aligned} \cos \alpha_1 &= \cos (\angle X''OX') \\ \cos \beta_1 &= \cos (\angle X''OY') \\ \cos \gamma_1 &= \cos (\angle X''OZ') \\ \cos \alpha_2 &= \cos (\angle Y''OX') \\ &\text{etc.} \end{aligned}\right\} \qquad (7)$$

Figure 2 gives the definitions of the angles on the right-hand side of equations (7). We then substitute the transformation equations (5) and (6) into the equations of the surfaces $P(x'', y'', z'')$. Thus we obtain the equations of the surfaces with respect to the origin of our original coordinate system (x, y, z).

Since the above matrix is orthogonal, the inverse transformations are easily obtained, the inverse matrix being equal to the transform of the matrix. Primes are used to represent translated coordinate systems, and double primes to represent translated and rotated coordinate systems.

In order to trace a ray through an optical system, we must have, for the surfaces involved, continuous equations (although the functions need not be single-valued) and continuous partial derivatives. This condition should not eliminate any optical systems that we may wish to study. We determine the equations of the original incident rays by the physical situation desired; e.g., the angles of incidence of the rays. Then we obtain the intersections of these rays with the first optical surface by solving equation (8), the parametric equations of the incident rays, simultaneously with the equation of the surface:

The relative index of refraction, n, is -1. The vector equation to be solved is then:

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \sin\gamma\cos a & \sin\gamma\sin a & \cos\gamma \\ 2x & 2y & -4f \end{vmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2x & 2y & -4f \\ R & S & T \end{vmatrix}, \tag{12}$$

where x and y are the points of intersection of the rays with the mirror. Since our system is a telescope, we can arbitrarily pick the points on the mirror to represent, for example, the extreme rays in the system. Equation 12 gives:

$$2f\sin\gamma\sin a + y\cos\gamma = -(2fS + yT) \tag{13}$$

or

$$S = -\frac{y(T + \cos\gamma) + 2f\sin\gamma\sin a}{2f}; \tag{13a}$$

$$2f\sin\gamma\cos a + x\cos\gamma = -(2fR + xT) \tag{14}$$

or

$$R = -\frac{x(T + \cos\gamma) + 2f\sin\gamma\cos a}{2f}; \tag{14a}$$

and

$$T^2 = 1 - \frac{1}{4f^2}\left[(x^2 + y^2)(T + \cos\gamma)^2 + 4f^2\sin^2\gamma + 4f\sin\gamma(T + \cos\gamma)(x\cos a + y\sin a)\right] \tag{15}$$

or

$$T^2\left(1 + \frac{x^2 + y^2}{4f^2}\right) + T\left[\frac{\cos\gamma(x^2 + y^2)}{2f^2} + \frac{\sin\gamma(x\cos a + y\sin a)}{f}\right]$$
$$+ \frac{\sin\gamma\cos\gamma(x\cos a + y\sin a)}{f} + \cos^2\gamma\left(\frac{x^2 + y^2}{4f^2} + 1\right) = 0. \tag{15a}$$

After obtaining T from this equation, we can readily solve for R and S. The reflected ray is then determined by its direction cosines and the point (x, y, z) on the mirror at which it was reflected.

For our own purposes, we place the faceplate of our television camera at the focus of this system. We will consider the reflected rays from the mirror to be the rays incident upon the front surface of the faceplate. The case of a double concave faceplate will be considered. Both surfaces of the faceplate are spherical, and its thickness at the center is t.

In this coordinate system, (x'', y'', z''), the equation of the first surface is:

$$x''^2 + y''^2 + z''^2 = R_1^2 . \tag{16}$$

The equation of the second surface is:

$$x''^2 + y''^2 + \left[z'' - (R_1 + R_2 + t)\right]^2 = R_2^2 . \tag{17}$$

We first translate these equations along the z-axis to a point, z'_o. Equations (16) and (17) then become:

$$x'^2 + y'^2 + (z' - z'_o)^2 = R_1^2 , \tag{16a}$$

$$x'^2 + y'^2 + \left[z' - (z'_o + R_1 + R_2 + t)\right]^2 = R_2^2 . \tag{17a}$$

Since the section of the paraboloid under consideration is not symmetrical with respect to the z-axis, we must also rotate the coordinate system at an angle δ in order to obtain the best focus. This angle is found by rotating the system back and forth in finite steps about an appropriately chosen initial angle until the best "spot size" is obtained over the entire field. The rotation is performed about the x-axis; therefore, $x = x'$. The resulting matrix equation is:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \cos\delta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} , \tag{18}$$

where the unprimed coordinates are in the system in which the equation of the paraboloid was written.

The final equations of the faceplate surface are:

$$x^2 + (y\cos\delta + z\sin\delta)^2 + (z\cos\delta - y\sin\delta - z_o)^2 = R_1^2 , \tag{16b}$$

$$x^2 + (y\cos\delta + z\sin\delta)^2 + \left[z\cos\delta - y\sin\delta - (z_o + R_1 + R_2 + t)\right]^2 = R_2^2 ; \tag{17b}$$

or

$$x^2 + y^2 + z^2 + z_o^2 + 2z_o(y \sin \delta - z \cos \delta) = R_1^2 , \tag{16c}$$

$$x^2 + y^2 + z^2 + k^2 + 2k(y \sin \delta - z \cos \delta) = R_2^2 , \tag{17c}$$

where

$$k = z_o + R_1 + R_2 + t .$$

The intersections of the ray reflected from the mirror and the first surface of the faceplate are obtained, and then the direction cosines of the reflected ray are found. The index of refraction of the faceplate must be used in this solution. The equations of the rays incident upon the second surface can be obtained from the direction cosines, and the intersections of the rays with the first surface. The final solution of the problem is the determination of the intersections of the rays with the second surface of the faceplate, since this represents physically the photoemissive surface of an ebicon television image tube.

Several square roots are involved in this solution. We chose their signs to give the physically correct answer; i.e., intersections with the equation within the range of values that actually represents the optical surface.

We have employed this method with the IBM-704 at the Computations and Analysis Division of the Smithsonian Astrophysical Observatory. For each angle of incidence we traced five rays through the system, one ray striking the center of the mirror and four at the edges, 90° apart (fig. 5). The computed spots were compared with the photographs taken with a plane image surface in our optics laboratory. We found that these rays gave both the size and shape of the image quite accurately. With a greater number of rays, we can determine an approximate intensity distribution within the image. After determining the angle δ, we found the best focus by moving the faceplate both toward and away from the mirror until the best possible spot size was obtained over the entire field.

We have used this same method to evaluate a Schwarzschild telescope system, a two-mirror system in which both reflecting surfaces are aspheric. We have also considered a spherical mirror used at grazing incidence, and a similar mirror for which a cubic correction factor had been introduced. These mirrors used at grazing incidence form the basic components of an x-ray telescope or microscope system.

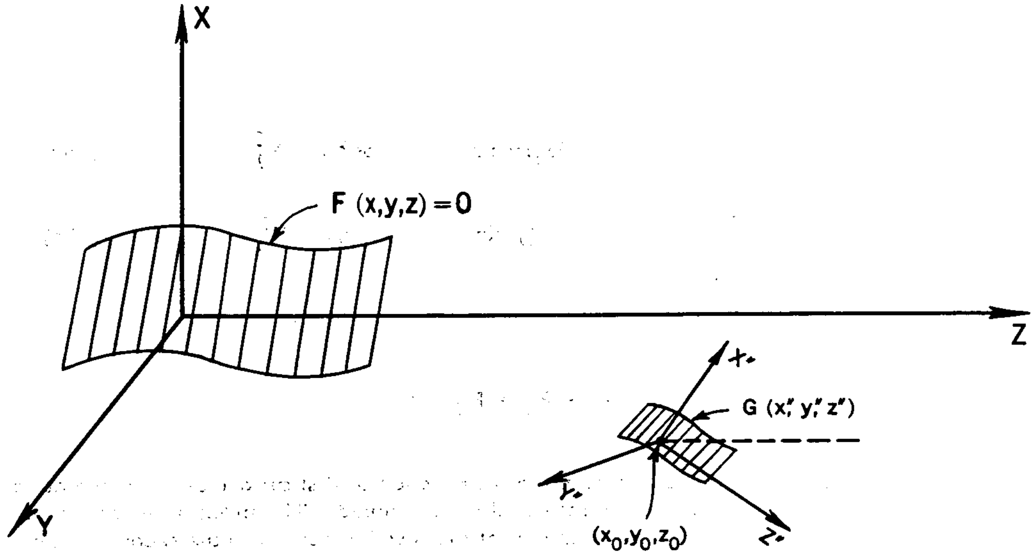

FIGURE 1. --Coordinate systems for (a set of) two optical surfaces.

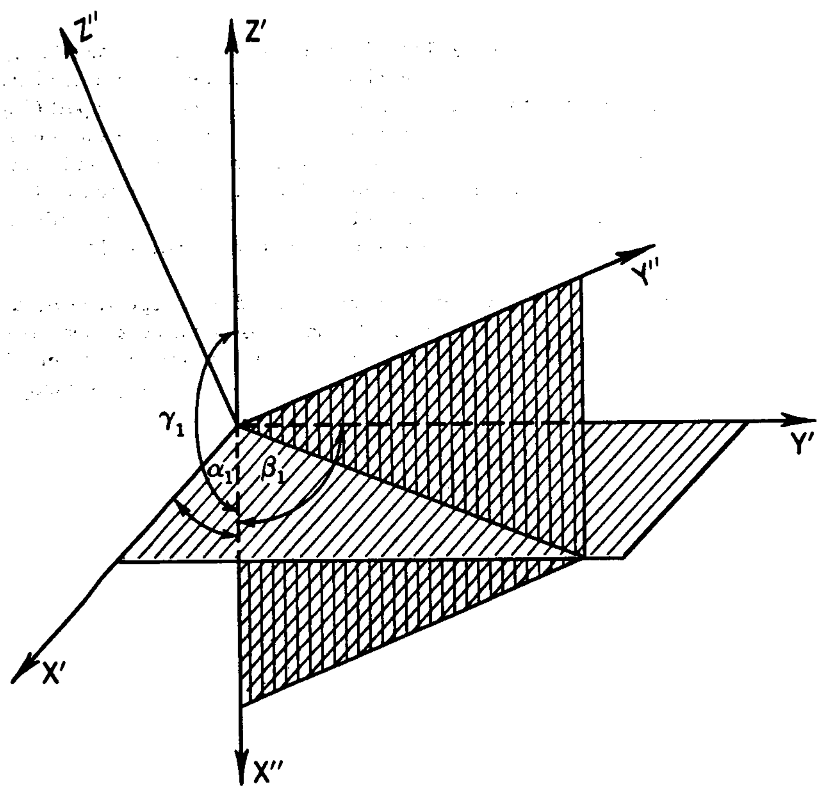

FIGURE 2. -- Geometrical illustration of the angles defined in equation (7).

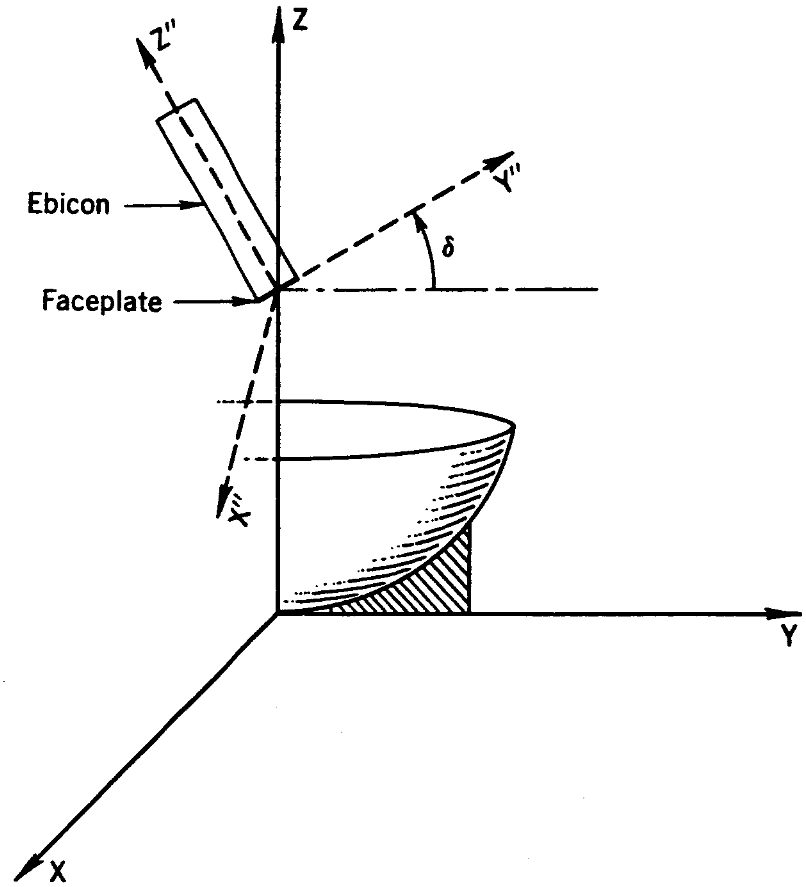

FIGURE 3. --Use of the off-axis paraboloid for a television telescope.

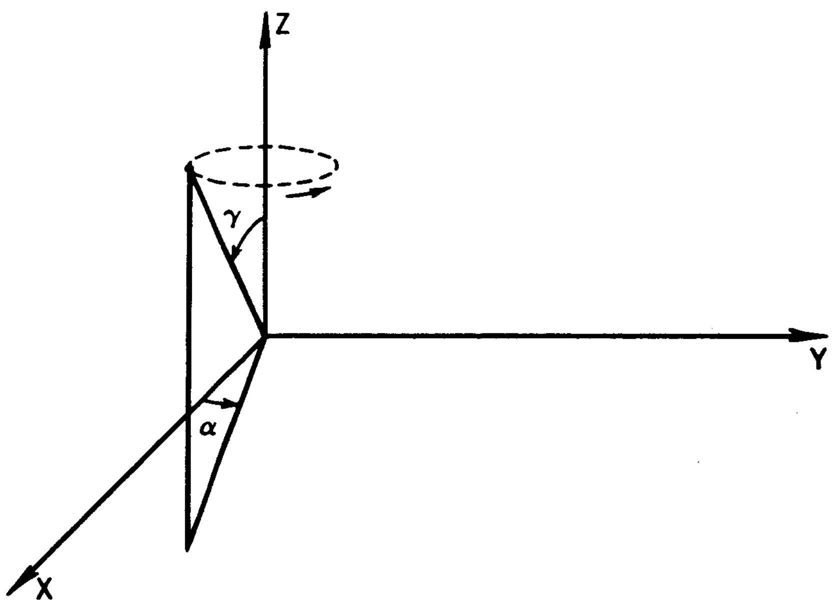

FIGURE 4. --Direction conventions for rotation of coordinate system.

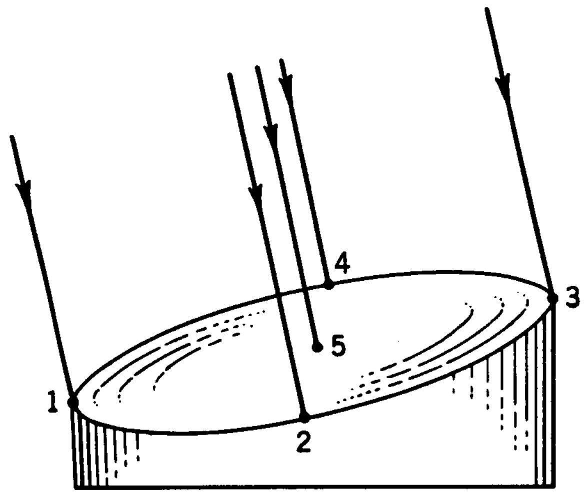

FIGURE 5. -- Nomenclature for numbering of rays.

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A DETERMINATION OF THE ELLIPTICITY OF THE EARTH'S EQUATOR FROM THE MOTION OF TWO SATELLITES

by

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p11-24 14 refs

Abstract -- The effect of the most important longitude-dependent term of the geopotential on the motion of artificial satellites is expressed by simple formulas. Its numerical coefficient β can be interpreted as the ellipticity of the earth's equator, the phase constant λ_X gives the geographical longitude of the semi-major axis. Based on photoreduced Baker-Nunn observations of the satellites 1959 α1 and 1959 Eta the values of these constants were determined to be

$$\beta = (3.21 \pm .29) \times 10^{-5} \quad \text{and} \quad \lambda_X = -33^\circ\!.15 \pm {}^\circ\!.53 \ .$$

Introduction

The primary mission of Satellite 1959 α1 (Vanguard II) was to scan the earth's cloud cover, and that of 1959 Eta (Vanguard III) to measure the earth's magnetic field. These experiments, however, could be conducted only for a limited period, as the transmitter of Vanguard II ceased operating 27 days after launching, and that of Vanguard III 85 days after launching. Since then the Smithsonian Astrophysical Observatory has had the responsibility for tracking both objects, and has obtained many thousands of optical observations of them by means of the Baker-Nunn camera stations and the voluntary Moonwatch teams. Because of their high quality, these observations became very valuable for different research projects, such as the study of the upper atmosphere, the determination of the earth's gravitational field, and other investigations now under way. For our present purpose, the determination of the coefficient and phase of the first longitude-dependent spherical harmonic in the geopotential, only the most accurate observations can be used. These are without question the positions obtained by the Photoreduction Section, at the Cambridge headquarters of the Smithsonian Astrophysical Observatory, from the Baker-Nunn photographs. For simplicity, we call these positions photoreduced observations as opposed to the field-reduced observations; the latter are the less accurate positions determined in the field, i.e., at any one of the 12 Baker-Nunn camera stations, immediately after a pass of the satellite over the station.

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The Geometrical Interpretation of the Geopotential

For easy reference we collect first some well known formulas related to the development of the geopotential of a rotating rigid body into a series of spherical harmonics. (According to the convention in theoretical physics, we are dealing here with a force function, which is the negative of potential energy.)

Let X, Y, Z be the coordinate system of the principal axes of inertia with the origin at the center of mass; and $A \leqq B \leqq C$ the moments of inertia about these axes. Then the gravitational potential of the body can be written in the form

$$U = \frac{\mu}{r}\left\{1 + \frac{A + B + C - 3I}{2m_e r^2}\right\} + 0\left(\frac{1}{r^4}\right) , \tag{1}$$

where $\mu = fm_e$ is the product of the constant of gravitation and the mass of the earth,

$$r^2 = X^2 + Y^2 + Z^2 ;$$

and

$$I = \frac{1}{r^2}(AX^2 + BY^2 + CZ^2)$$

is the moment of inertia about the line joining the center of mass to the point considered. Expression (1) is sometimes referred to as MacCullagh's formula (Jeffreys, 1959).

Let us introduce spherical harmonics in the usual way by the relations

$$Z^2 - \frac{X^2 + Y^2}{2} = r^2\, P_2(\sin\Phi) = \frac{r^2}{2}(3\sin^2\Phi - 1) , \tag{2}$$

$$3(X^2 - Y^2) = r^2\, P_2^{(2)}(\sin\Phi)\,\cos 2\Lambda = 3r^2\cos^2\Phi\,\cos 2\Lambda , \tag{3}$$

where Φ denotes the latitude; and Λ the longitude reckoned from the principal axis X; $P_2(\sin\Phi)$ is a Legendre polynomial; and

$$P_2^{(2)}(\sin\Phi) = \frac{\cos^2\Phi\, d^2 P_2(\sin\Phi)}{(d\sin\Phi)^2} .$$

The polynomial (2) is called a zonal solid spherical harmonic; the polynomial (3), on the other hand, is called a sectorial solid spherical harmonic. Using the notations

$$J_2 = \frac{2C - (A + B)}{2m_e a_e^2} \quad \text{and} \quad J_2^{(2)} = \frac{A - B}{4m_e a_e^2},$$

we can write

$$U = \frac{\mu}{r}\left\{1 - J_2\left(\frac{a_e}{r}\right)^2 P_2(\sin \Phi) - J_2^{(2)}\left(\frac{a_e}{r}\right)^2 P_2^{(2)}(\sin \Phi) \cos 2\Lambda\right\}. \tag{4}$$

Because all gravity measurements are made with respect to a frame rotating with the earth, the geopotential Ψ is the sum of the gravitational potential U and the potential of the centrifugal force; that is,

$$\Psi = U + \frac{\dot{\psi}^2}{2}(X^2 + Y^2), \tag{5}$$

where $\dot{\psi}$ is the angular velocity of the earth's rotation.

To express the constants J_2 and $J_2^{(2)}$ by geometrical quantities related to the shape of the earth, and by a quantity related to $\dot{\psi}$, we assume that the surface of the earth is very nearly an equipotential surface of the geopotential Ψ. This hypothesis of hydrostatic equilibrium has certain cosmogonical consequences. It must be kept in mind, however, that even in the case of the earth, it gives only a good approximation to the real situation; and in the case of the moon it fails completely. Be that as it may, we suppose that

$$\Psi = \text{const} \tag{6}$$

on the surface of a triaxial ellipsoid, which deviates but slightly from a sphere; that surface will be considered as the mean surface of the earth. In the equation

$$\frac{X^2}{a_e^2} + \frac{Y^2}{b_e^2} + \frac{Z^2}{c_e^2} = 1 \tag{7}$$

of this geoid, we put

$$b_e = a_e(1 - \beta), \quad c_e = a_e(1 - \gamma), \tag{8}$$

and call β the ellipticity of the earth's equator, and γ the oblateness of the earth. Also, we introduce the small dimensionless quantity

$$\alpha = \frac{\psi^2}{\mu/a_e^3} , \tag{9}$$

which is the ratio of the squares of the earth's angular velocity and the Keplerian mean motion of a fictitious satellite with semimajor axis $a = a_e$.

We will need the expansion of a_e/r on the surface of the geoid in spherical harmonics. From equations (7) and (8) it follows that to the first order in the small quantities β and γ

$$(1 - 2\gamma) \frac{X^2}{a_e^2} + (1 - 2\gamma + 2\beta) \frac{Y^2}{a_e^2} + Z^2 = 1 - 2\gamma ;$$

hence

$$\frac{a_e}{r} = (1 + \gamma) - \gamma \frac{(X^2 + Y^2)}{a_e^2} + \beta \frac{Y^2}{a_e^2} .$$

But as a consequence of equations (2) and (3) we have

$$X^2 + Y^2 = \frac{2r^2}{3} \left\{1 - P_2 (\sin \Phi)\right\} ,$$

$$Y^2 = \frac{r^2}{6} \left\{2 - 2P_2(\sin \Phi) - P_2^{(2)}(\sin \Phi) \cos 2\Lambda\right\}, \tag{10}$$

so that expressed in spherical harmonics

$$\frac{a_e}{r} = (1 + \frac{\beta + \gamma}{3}) + \frac{2\gamma - \beta}{3} P_2 (\sin \Phi) - \frac{\beta}{6} P_2^{(2)} (\sin \Phi) \cos 2\Lambda . \tag{11}$$

Then a comparison of equations (4), (5), (9), and (11) shows that on the surface of the geoid

$$\Psi = \frac{\mu}{a_e} \left\{\left(1 + \frac{\gamma+\beta+\alpha}{3}\right) + \left(\frac{2\gamma-\beta-\alpha}{3} - J_2\right) P_2(\sin \Phi) - \left(\frac{\beta}{6} + J_2^{(2)}\right) P_2^{(2)}(\sin \Phi) \cos 2\Lambda\right\},$$

again to the first order in the small quantities α, β, γ . Because the development of a function of the angles Φ and Λ into a series of spherical surface harmonics is unique, the condition (6) amounts to the validity of the relations

$$J_2 = \frac{2\gamma - (\beta + \alpha)}{3} \quad \text{and} \quad J_2^{(2)} = -\frac{\beta}{6} \tag{12}$$

among the coefficients J_2 and $J_2^{(2)}$ on the one hand, and the constants α, β, γ on the other.

Geodesists are usually concerned with the gravity, g, which is the absolute value of the gradient $\nabla\Psi$. Neglecting quantities of the second order in α, β, and γ, one first obtains

$$|\nabla\Psi| = \left|\frac{\partial\Psi}{\partial r}\right| + \dots = \frac{\mu}{a_e^2}\left\{\left(\frac{a_e}{r}\right)^2 - 3J_2\left(\frac{a_e}{r}\right)^4 P_2(\sin\Phi)\right.$$

$$\left. - 3J_2^{(2)}\left(\frac{a_e}{r}\right)^4 P_2^{(2)}(\sin\Phi)\cos 2\Lambda - \frac{2\alpha}{3}\left(\frac{r}{a_e}\right)\left[1 - P_2(\sin\Phi)\right]\right\} .$$

On the surface of the earth, this expression becomes

$$g = g_e\left\{1 + \frac{5\alpha + \beta - 2\gamma}{2}\sin^2\Phi + \frac{\beta}{2}\cos^2\Phi\cos 2\Lambda\right\}, \tag{13}$$

with

$$g_e = \frac{\mu}{a_e^2}\left(1 - \frac{3\alpha - \beta - 2\gamma}{2}\right) ,$$

when use is made of relations (12)

The value of the constant α can be determined with high accuracy. A sidereal day equals 86164.10 seconds; therefore, $\psi = 7.292115 \times 10^{-5}\ \text{sec}^{-1}$. Then, adopting from O'Keefe, Eckels and Squires (1959) the value

$$\mu = 3.98618 \times 10^5\ \text{km}^3\ \text{sec}^{-2}$$

and the value

$$a_e = 6.378388 \times 10^3\ \text{km}$$

of the international ellipsoid, we obtain from relation (9)

$$a = 3.46165 \times 10^{-3} .$$

The most accurate value of the coefficient J_2 is probably that of Kozai (1961), who has found that

$$J_2 = (1.08219 \pm .00002) \times 10^{-3} . \tag{14}$$

To take full advantage of his determination's accuracy, we must consider the second-order expression of J_2, instead of the relation given by the first equation in (12). Such an expression can be derived by a straight-forward, though more laborious extension of the argument that led to relations (12). The result is

$$J_2 = \frac{2\gamma - (a + \beta)}{3} - \frac{\gamma^2}{3} + \frac{3a\gamma}{7} . \tag{15}$$

Here in the second-order terms it is sufficient to use for the constant γ an approximate value, $1/298.3 = 3.352 \times 10^{-3}$, which together with the above value of a gives

$$J_2 = \frac{2\gamma - (a + \beta)}{3} + 1.225 \times 10^{-6} .$$

The determination of the oblateness γ from the coefficient J_2 is sensibly affected by the value of the ellipticity β, as shown in the following table:

when β	then γ
0	$3.35227 \times 10^{-3} = 1/298.305$
2×10^{-5}	$3.36227 \times 10^{-3} = 1/297.418$
4×10^{-5}	$3.37227 \times 10^{-3} = 1/296.536$
6×10^{-5}	$3.38227 \times 10^{-3} = 1/295.659$.

The data on the ellipticity of the equator are rather contradictory in the literature. They were determined by analyzing the measures of gravity over extended areas of the earth, and seem to be influenced significantly by the distribution of the observational material. For instance, from the treatise of Heiskanen and Meinesz (1958) we list these data:

	β	λ_X
Helmert, 1915	3.6×10^{-5}	-17°
Heiskanen, 1924	5.4×10^{-5}	$+18^\circ$
Heiskanen, 1928	3.8×10^{-5}	0°
Niskanen, 1945	4.6×10^{-5}	-4°
Uotila, 1957	2.1×10^{-5}	-6°,

where λ_X is the geographic longitude of the direction of the principal axis X, so that for any longitude λ we have

$$\Lambda = \lambda - \lambda_X .$$

According to Jeffreys (1959), however,

$$\beta = (2.5 \pm .9) \times 10^{-5} , \quad \lambda = 0^\circ .$$

In his textbook on celestial mechanics, Subbotin (1949) adopted the value

$$\beta = 1/30000 = 3.3 \times 10^{-5} .$$

In his detailed study, Jongolovich (1957) derives the equivalent of the values

$$\beta = 3.57 \times 10^{-5} , \ \lambda_X = -7.^\circ 7 .$$

From a recent paper by Kaula (1959) one can infer the values

$$\beta = 3.67 \times 10^{-6} , \ \lambda_X = -20.^\circ 8 .$$

The Effect of the Ellipticity of the Earth's Equator on the Motion of Satellites.

Several investigators have pointed out that the ellipticity of the earth's equator might have a detectable effect on the motion of artificial satellites, and therefore a careful analysis of accurate satellite observations could lend itself to a better determination of the constants β and λ_X . In this connection let us refer to the papers by O'Keefe and Batchlor (1957); Robe (1959); Cook (1960); Sehnal (1960); and Musen (1960).

To derive the perturbations caused by the presence of the sectorial harmonic in question, we can confine ourselves to a very simple theory. The disturbing function, which is

$$R = \frac{\beta}{2} \frac{\mu}{a_e} \left(\frac{a_e}{r}\right)^3 \cos^2\Phi \cos 2\Lambda ,$$

must first be transformed into a function of the orbital elements a or n, e, I, M, ω, and Ω of the satellite. Let

v = the true anomaly,
u = v + ω, the argument of latitude,
w = the projection of u on the equator, that is the longitude of the satellite reckoned from the ascending node of its orbit,

θ_G and θ_X = the mean sidereal time at Greenwich and at the intersection of the principal axis X with the surface of the earth.

Note that

$$\Lambda = w - (\theta_X - \Omega) \quad ,$$

and

$$\lambda_X = \theta_X - \theta_G \; .$$

The angles Φ, w are related to the angles I, u by the formulas

$$\cos \Phi \cos w = \cos u$$

and

$$\cos \Phi \sin w = \cos I \sin u \; ,$$

in accordance with spherical trigonometry. These equations can be written in the complex form

$$\cos \Phi \exp(iw) = \cos u + i \cos I \sin u = \cos^2 \frac{I}{2} \exp(iu) + \sin^2 \frac{I}{2} \exp(-iu) \; ,$$

which immediately yields

$$\cos \Phi \exp(i\Lambda) = \cos^2 \frac{I}{2} \exp\left[i(u - \theta_X + \Omega)\right] + \sin^2 \frac{I}{2} \exp\left[-i(u + \theta_X - \Omega)\right] \; .$$

But

$$\cos^2 \Phi \cos 2\Lambda = \mathrm{Re}\left\{\left[\cos \Phi \exp(i\Lambda)\right]^2\right\} \; ,$$

so that introducing the abbreviations

$$c = \cos I, \quad s = \sin I, \quad n^2 = \frac{\mu}{a^3}$$

we obtain the disturbing function in the form

$$R = \frac{\beta a_e^2}{8} n^2 \left(\frac{a}{r}\right)^3 \left\{ 2s^2 \cos 2(\theta_X - \Omega) + (1 + c)^2 \cos 2(u - \theta_X + \Omega) + (1 - c)^2 \cos 2(u + \theta_X - \Omega) \right\} .$$

General formulas for the disturbing function of any tesseral harmonic can be found in a paper by Groves (1960).

In our problem it is not necessary to carry out the development of this function into a Fourier series in terms of the mean anomaly M, because the contribution of the short-periodic terms to the perturbations turns out to be negligibly small. As the Fourier series of the functions

$$\left(\frac{a}{r}\right)^3 \cos 2v \quad \text{and} \quad \left(\frac{a}{r}\right)^3 \sin 2v$$

do not contain constant terms, and the constant term

$$\text{in} \quad \left(\frac{a}{r}\right)^3 \quad \text{is} \quad (1 - e^2)^{-3/2} ,$$

the "long-periodic" part of R becomes simply

$$\bar{R} = \left(\frac{\beta}{4}\right) a_e^2 n^2 (1 - e^2)^{-3/2} s^2 \cos 2(\theta_X - \Omega) . \tag{16}$$

The perturbations caused by the oblateness of the earth concern us here only to the extent that Ω is not constant, but rather a linear function of time

$$\Omega = \Omega_o + \Omega_1 (t - t_o) ,$$

so that

$$\theta_X - \Omega = \theta_{X_o} - \Omega_o + (\theta_1 - \Omega_1)(t - t_o) . \tag{17}$$

Substituting function (16) into the differential equations of the orbital elements (see, for example, Moulton (1959)) and integrating with respect to time, we obtain at once

$$\delta\omega = \beta \frac{na_e^2 (3 - 5c^2)}{8 (\theta_1 - \Omega_1) p^2} \sin 2(\theta_X - \Omega) \quad ,$$

$$\delta\Omega = \beta \frac{na_e^2 c}{4(\theta_1 - \Omega_1) p^2} \sin 2(\theta_X - \Omega) \quad ,$$

$$\delta I = \beta \frac{na_e^2 s}{4(\theta_1 - \Omega_1) p^2} \cos 2(\theta_X - \Omega) \quad , \tag{18}$$

$$\delta M = \beta \frac{3na_e^2 (1 - e^2)^{1/2} s^2}{8(\theta_1 - \Omega_1) p^2} \sin 2(\theta_X - \Omega) \, .$$

The semimajor axis a and the eccentricity e do not undergo long-periodic perturbations due to the ellipticity of the equator. The question now is how to use these formulas for the determination of the constant β, and whether the accuracy of the observations made with the Baker-Nunn cameras is sufficient for such a determination.

The Computation of the Constants β and λ_X

In the Differential Orbit Improvement Program of the Smithsonian Astrophysical Observatory developed for the IBM-704 computer by G. Veis and Ch. H. Moore, the orbital elements ω, Ω, I, e, M can be represented as polynomials up to the 7th degree, plus sine-terms and exponential functions of the time t. As many as 23 of the data figuring in such a representation can be varied in an iterative least-squares solution, but we usually limit ourselves to much less unknown in order to obtain meaningful results. It takes some experience to decide properly which quantities should be varied in a particular application.

The numerical material for the present investigation was obtained in December, 1960 by several consecutive runs of this program, as applied to 187 photoreduced Baker-Nunn observations of Satellite 1959 α 1 covering the interval April 6 through 25, 1960; and to 216 observations of Satellite 1959 Eta covering the interval May 8 through 27, 1960. The use of two satellites is not a theoretical necessity, but a precautionary measure in order to give two independent determinations of the constants in question. While it is not possible here to furnish all the details of these orbit computations, their important features can be summarized as follows. Because the phase at which the trigonometric terms (18) are "observed" at a particular station varies only slightly from day to day, it is desirable to use a long interval of time for the computation of the constants β and λ_X. However, even in the case of such relatively high orbiting satellites as 1959 α 1 and 1959 Eta, the effect of the air drag cannot be described with sufficient accuracy in too long an interval of time by simple polynomial expressions. As a compromise, some numerical experimentation led to the 20-day intervals noted above, with third-degree polynomials in the mean anomaly M. The best possible polynomial representation of all the orbital elements was determined, treating ω and Ω as polynomials of the second degree, I and e as polynomials of the first degree, and M as a polynomial of the third degree. This means that the long-range direct and indirect effects of the air drag, of the radiation pressure, of secular and long-periodic perturbations due to the even and odd zonal harmonics in the earth's gravitational potential, and of luni-solar perturbations were treated in an empirical way. Analytical expressions for the first-order short-periodic perturbations due to the oblateness of the earth are incorporated in the computer program. The coefficients of these polynomials were improved by including the trigonometric terms (18) into the orbital elements with the constants

$$\beta = 3.8 \times 10^{-5} \quad \text{and} \quad \lambda_X = 0^\circ$$

of Heiskanen. In several consecutive runs these constants were then treated as unknowns only in the right ascension of the node Ω, because one would expect no short-periodic air drag effects in this orbital element, and because the amplitude of $\delta\Omega$ turned out to be the biggest one. The improved values of β and λ_X were substituted into $\delta\omega$, $\delta\Omega$, δI, δM, and then the computation of these constants was repeated until the new values agreed with the old ones within their standard errors.

Here are the relevant results of these computations

	Satellite	
	1959 α1	1959 Eta
t_o	April 16.0, 1960	May 19.0, 1960
θ_{Go}	204°.1383	236°.6646
θ_1	1.002738 rev/day	
Ω_o	138°.0499	155°.1312
Ω_1	-3°.51067/day	-3°.27567/day
n	11.463937 rev/day	11.069014 rev/day
a	1.302343 a_e	1.333153 a_e
e	0.1646379	0.1891083
$q = a(1-e)-a_e$	561 km	517 km
I	32°.8796	33°.3569 .

Consequently, expressions (18) become

$\delta\omega$	$-\beta \times 0.4641 \sin 2(\theta_X - \Omega)$	$-\beta \times 0.4041 \sin 2(\theta_X - \Omega)$
$\delta\Omega$	$\beta \times 1.4808 \sin 2(\theta_X - \Omega)$	$\beta \times 1.3824 \sin 2(\theta_X - \Omega)$
δI	$\beta \times 0.9572 \cos 2(\theta_X - \Omega)$	$\beta \times 0.9100 \cos 2(\theta_X - \Omega)$
δM	$\beta \times 0.7688 \sin 2(\theta_X - \Omega)$	$\beta \times 0.7370 \sin 2(\theta_X - \Omega)$.

In the course of the trigonometric least-squares fit to the observations, the IBM-704 electronic computer actually obtained the amplitude of $\delta\Omega$ and twice the phase angle $\theta_{Xo} - \Omega_o = \lambda_X + (\theta_{Go} - \Omega_o)$. From these data we immediately obtain

β	$(3.47 \pm .28) \times 10^{-5}$	$(2.88 \pm .32) \times 10^{-5}$
λ_X	$-33^\circ\!.5 \pm 1^\circ\!.9$	$-32^\circ\!.3 \pm 3^\circ\!.1$.

The agreement between the two sets of data seems reasonable; taking the weighted mean values, we obtain

$$\beta = (3.21 \pm .29) \times 10^{-5}$$

and

$$\lambda_X = -33^\circ\!.15 \pm {}^\circ\!.53 \quad .$$

This numerical value of the ellipticity gives the perturbation $\delta\omega$, . . . expressed in radians. For the use of other angular units we note that

$$\beta = 5.11 \times 10^{-6} \text{ rev} = 1^\circ\!.84 \times 10^{-3} = 6''\!.62.$$

According to definition (8) of the quantity β, the difference between the axes a_e and b_e is 205 m; as a consequence of equation (15), the oblateness becomes

$$\gamma = 3.36832 \times 10^{-3} = 1/296.884 \; .$$

Needless to say, the above determination is far from being a final one. It will be improved as soon as more accurate coordinates of the Baker-Nunn camera stations are available.

References

COOK, A.H.

1960. Papers presented to the 1st International Symposium on Space Sciences in Nice.

GROVES, G.V.

1960. Motion of a satellite in the earth's gravitational field. Proc. Roy. Soc., vol. 254A, pp. 48-65.

HEISKANEN, W.A. and MEINESZ, F.A.

1958. The earth and its gravity field. McGraw-Hill Co., New York, p. 79.

JEFFREYS, H.

1959. The earth. Cambridge University Press, Cambridge, 4th ed., chaps. IV and V.

JONGOLOVICH, I.D.

1957. Gravitational potential of the earth. Bull. Inst. Theor. Astron., vol. 6, pp. 505-523.

KAULA, W.M.

1959. Statistical and harmonic analysis of gravity. Journ. Geophys. Res., vol. 64, pp. 2401-2421.

KOZAI, Y.

1961. The gravitational field of the earth from the motion of three satellites. Astron. Journ., in press.

MOULTON, F.R.

1959. An introduction to celestial mechanics. Macmillan Co., New York, new ed., p. 399.

MUSEN, P.

1960. On the motion of a satellite in an asymmetrical gravitational field. Journ. Geophys. Res., vol. 65, pp. 2783-2792.

O'KEEFE, J.A. and BATCHLOR, C.D.

1957. Perturbations of a close satellite by the equatorial ellipticity of the earth. Astron. Journ., vol. 62, pp. 183-185.

O'KEEFE, J.A., ECKELS, A., and SQUIRES, R.K.

1959. The gravitational field of the earth. Astron. Journ., vol. 64, pp. 245-253.

ROBE, H.

1959. Étude de l'orbite équatoriale d'un satellite artificiel dans l'hypothèse d'un ellipsoïde terrestre à 3 axes inégaux. Bull. Soc. Roy. Sci. Liège, vol. 28, pp. 207-221.

SEHNAL, L.
1960. The influence of the equatorial ellipticity of the earth gravitational field on the motion of a close satellite. Bull. Astron. Inst. Czechoslovakia, vol. 11, pp. 90-93.

SUBBOTIN, M.F.
1949. Course of celestial mechanics III. Gostehizdat, p. 278.

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EFFECTS OF SOLAR RADIATION PRESSURE ON THE MOTION OF AN ARTIFICIAL SATELLITE

by

Yoshihide Kozai[1] p 25-34 5 refs

The effects of solar radiation pressure on the motion of an artificial satellite have been studied by several authors (Musen, 1960; Parkinson, Jones, and Shapiro, 1960). As they predicted, the orbit of Satellite 1960 ι1 (Echo I) has been greatly affected by the solar radiation pressure (Shapiro and Jones, 1960). And even for Satellite 1958 β2 (Vanguard I), which is of moderate size, Musen, Bryant, and Bailie (1960) found that the discrepancy between the observed and computed values of perigee height could be explained by the solar radiation effect.

When the author (Kozai, 1959; 1961) derived several constants of the earth's gravitational potential from the motion of artificial satellites, he did not take this effect into consideration. Although the effect is very small for the average satellite, it must be considered in the future in the reduction of observations.

In the present paper the author wants to study this problem in order to reduce the observations of satellites of moderate size. The analytical expressions for the perturbations of the first order are easily obtained; however, the two limits of integration are derived by numerical methods.

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Disturbing Functions

The equations of variations to be solved are the following:

$$\frac{da}{dt} = \frac{2na^3}{\sqrt{1 - e^2}} F\left\{S(v) \sin v + T(v) \frac{p}{r}\right\},$$

$$\frac{de}{dt} = na^2\sqrt{1 - e^2}\, F\left[S(v) \sin v + T(v)\left\{\cos v + \frac{1}{e}\left(1 - \frac{r}{a}\right)\right\}\right],$$

$$\frac{di}{dt} = \frac{na^2}{\sqrt{1 - e^2}} W F \frac{r}{a} \cos L\,,$$

$$\sin i \frac{d\Omega}{dt} = \frac{na^2}{\sqrt{1 - e^2}} W F \frac{r}{a} \sin L\,, \tag{1}$$

$$\frac{d\omega}{dt} = -\cos i \frac{d\Omega}{dt} + na^2 \frac{\sqrt{1 - e^2}}{e} F\left[-S(v) \cos v + T(v)\left(1 + \frac{r}{p}\right)\sin v\right],$$

$$\frac{dM}{dt} = n - 2a^2 F S(v) \frac{r}{a} - \sqrt{1 - e^2}\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right),$$

where the conventional notations are used for the orbital elements,

$$L = v + \omega\,,$$

$$p = a(1 - e^2)\,;$$

$n^2a^3FS(v)$, $n^2a^3FT(v)$, and n^2a^3FW are three components of the disturbing force due to the solar radiation pressure in the direction of the radius vector of the satellite, in the direction perpendicular to it in the orbital plane, and in the normal to the orbital plane; and F is a product of the mass area ratio, solar radiation pressure, and a reciprocal of GM.

To derive the expressions of $S(v)$, $T(v)$, and W, we make the following assumptions:

1) The distance of the sun and the satellite is infinite; that is, the parallax of the sun is negligible.

2) The solar flux is constant along the orbit of the satellite if there is no shadow.

3) There is no re-radiation from the surface of the earth.

Then the expressions are the following:

$$
\begin{aligned}
S(v) = &- \cos^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(\lambda_{\odot} - L - \Omega) \\
&- \sin^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(\lambda_{\odot} + \Omega - L) \\
&- \frac{1}{2}\sin i \sin \epsilon\left\{\cos(\lambda_{\odot} - L) - \cos(-\lambda_{\odot} - L)\right\} \\
&- \sin^2\frac{i}{2}\cos^2\frac{\epsilon}{2}\cos(\Omega - \lambda_{\odot} - L) \\
&- \cos^2\frac{i}{2}\sin^2\frac{\epsilon}{2}\cos(-\lambda_{\odot} - L - \Omega)\,,
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
W = &\sin i \cos^2\frac{\epsilon}{2}\sin(\lambda_{\odot} - \Omega) - \sin i \sin^2\frac{\epsilon}{2}\sin(\lambda_{\odot} + \Omega) \\
&- \cos i \sin \epsilon \sin \lambda_{\odot}\,,
\end{aligned}
$$

where $\lambda_{\odot}$ is the longitude of the sun, and ϵ is the obliquity. The expression of $T(v)$ is obtained if cos in $S(v)$ is replaced by sin except for the trigonometrical terms with an argument i, ϵ, $i/2$, or $\epsilon/2$.

Solutions

There is a very important difference between the solar radiation force and the gravitational force. The solar radiation force is sometimes a discontinuous function of time because when the satellite enters the shadow, its motion is free from this effect.

Suppose that the satellite exits from the shadow at a point where the corresponding eccentric anomaly of the satellite is E_1, and enters the shadow at E_2. If the force is continuous, the integral of the short-periodic effect can be neglected. However, for this case there is a possibility that this effect will be cumulative to a certain amount during a long interval of time. Therefore, the short-periodic terms must be kept in the solutions.

By use of the eccentric anomaly E as the independent variable, the perturbations of the first order after one revolution can be derived in closed forms as follows:

$$
\begin{aligned}
\delta a &= 2a^3 F \left| \left(S \cos E + T \sqrt{1 - e^2} \sin E \right) \right|_{E_1}^{E_2} , \\
\delta e &= a^2 F \sqrt{1 - e^2} \Bigg[\Bigg| \frac{1}{4} S \sqrt{1 - e^2} \cos 2E \\
&\qquad + T \left(-2e \sin E + \frac{1}{4} \sin 2E \right) \Bigg|_{E_1}^{E_2} \\
&\qquad + \frac{3}{2} \int T dE \Bigg] , \\
\delta i &= a^2 F \frac{W}{\sqrt{1 - e^2}} \Bigg[\Bigg| \left\{ \left(1 + e^2\right) \sin E - \frac{e}{4} \sin 2E \right\} \cos \omega \\
&\qquad + \sqrt{1 - e^2} \left(\cos E - \frac{e}{4} \cos 2E \right) \sin \omega \Bigg|_{E_1}^{E_2} \\
&\qquad - \frac{3}{2} e \int W \cos \omega \, dE \Bigg] , \\
\sin i \; \delta \Omega &= a^2 F \frac{W}{\sqrt{1 - e^2}} \Bigg[\Bigg| \left\{ \left(1 + e^2\right) \sin E - \frac{e}{4} \sin 2E \right\} \sin \omega \\
&\qquad - \sqrt{1 - e^2} \left(\cos E - \frac{e}{4} \cos 2E \right) \cos \omega \Bigg|_{E_1}^{E_2} \\
&\qquad - \frac{3}{2} e \int W \sin \omega \, dE \Bigg] , \\
\delta \omega &= - \cos i \; \delta \Omega + a^2 F \frac{\sqrt{1 - e^2}}{e} \Bigg[\Bigg| S \left(e \sin E + \frac{1}{4} \sin 2E \right) \\
&\qquad + T \sqrt{1 - e^2} \left(e \cos E - \frac{1}{4} \cos 2E \right) \Bigg|_{E_1}^{E_2} \\
&\qquad - \frac{3}{2} \int S dE \Bigg] ,
\end{aligned}
\tag{3}
$$

$$\delta M = -\frac{3}{2}\int_0^{2\pi} \frac{\delta a}{a}\, dM - \sqrt{1 - e^2}\,\delta\omega - \sqrt{1 - e^2}\cos i\,\delta\Omega$$

$$- 2a^2F\Bigg[\Big|S\Big\{(1 + e^2)\sin E - \frac{e}{4}\sin 2E\Big\}$$

$$- T\sqrt{1 - e^2}\,(\cos E - \frac{e}{4}\cos 2E)\Big|_{E_1}^{E_2}$$

$$- \frac{3}{2}e\int S dE\Bigg] ,$$

where the limits of integration are E_1 and E_2 unless other values are written; S and T are the expressions of S(v) and T(v), in which L is replaced by ω; that is,

$$S = S(0) ,$$
$$T = T(0) . \tag{4}$$

If the satellite does not enter the shadow during one revolution, the terms depending explicitly on E vanish, and in particular, δa vanishes.

In the expressions of $\delta\omega$ and $\delta\Omega$, indirect effects of the solar radiation pressure through $\dot{\omega}$ and $\dot{\Omega}$ must be considered as follows:

$$\frac{d\delta\omega}{dt} = \frac{d\dot{\omega}}{de}\delta e + \frac{d\dot{\omega}}{di}\delta i + \frac{d\dot{\omega}}{da}\delta a ,$$

$$\frac{d\delta\Omega}{dt} = \frac{d\dot{\Omega}}{de}\delta e + \frac{d\dot{\Omega}}{di}\delta i + \frac{d\dot{\Omega}}{da}\delta a . \tag{5}$$

Shadow Equation

If the geocentric angular distance between the sun and the satellite be denoted by α, the shadow boundary is expressed by the equation,

$$r \sin \alpha = a_{\bullet} , \tag{6}$$

where $a_{\oplus}$ is the radius of the earth and is assumed to be constant. The following relations hold among r/a, E, S, T and α:

$$\frac{r}{a}\cos\alpha = -S(\cos E - e) - T\sqrt{1 - e^2}\sin E,$$

$$\frac{r}{a} = 1 - e\cos E. \tag{7}$$

If ω, Ω, and $\lambda_{\oplus}$ are assumed to be constant during one revolution of the satellite, S, T and e can also be regarded as constant; then E can be derived from equation (6).

However, as the equation is of the fourth degree with respect to sin E or cos E, it is very difficult to get general analytical solutions except for a circular orbit. If the orbit is circular, the terms of odd powers of sin E or cos E disappear, and the equation becomes quadratic for $\cos^2 E$ or $\sin^2 E$. And it is almost impossible to expand the solutions into power series of the eccentricity because of slow convergency. Even if the series is convergent, it is possible that the equation has no real root even though the equation has two real roots for the circular orbit, and vice versa.

Therefore, the author thinks that equation (7) must be solved numerically for every revolution. There are four roots; however, under a condition that $\cos\alpha$ must be negative, the number reduces to two, at most. If there is no real root, the satellite does not enter the shadow. And if there is only one real root, the satellite touches the shadow at one point.

Numerical Examples

The author has devised a program to calculate these effects on the IBM-704 computer. This program computes the inequalities of the orbital elements due to the solar radiation pressure, if the approximate expressions of the orbital elements are known.

As an example, inequalities of the orbital elements for Satellite 1958 $\beta 2$ during the period 36526-36615 (Modified Julian Days) are plotted in Figure 1. This computation is based on an estimated acceleration of 9.7×10^{-6} cm/sec^2 (Musen, Bryant, and Bailie, 1960). The semimajor axis is expressed in earth equatorial radii, and the mean anomaly is in revolutions. The author (Kozai, 1961) used earlier the same observations to derive geodetic constants; the effect of the solar radiation pressure was partly taken into account in that reduction.

The perturbations for Satellite 1960 $\iota 1$ have been roughly computed by this program, and have been checked with observed values. As the eccentricity is small (order of 10^{-2}), the change of the argument of perigee is very rapid due to the solar radiation. For this case it is better to use

$$\xi = e\sin\omega,$$

$$\eta = e\cos\omega,$$

instead of e and ω.

The following expressions are derived under an assumption that the satellite never enters the shadow, and they are compared with observations in Figure 2:

$$
\begin{aligned}
\xi &= \xi' + 0.00066 \\
\xi' &= 0.04409 \sin(91^\circ\!.99 + 2^\circ\!.965\, t) \\
&\quad + 0.03579 \sin(\lambda_\bullet - \Omega) + 0.00188 \sin(\lambda_\bullet + \Omega) \\
&\quad - 0.00519 \sin \lambda_\bullet , \qquad\qquad (8) \\
\eta &= 0.04409 \cos(91^\circ\!.99 + 2^\circ\!.965\, t) \\
&\quad + 0.03373 \cos(\lambda_\bullet - \Omega) - 0.00200 \cos(\lambda_\bullet + \Omega) \\
&\quad - 0.00169 \cos \lambda_\bullet ,
\end{aligned}
$$

where 0.00066 in ξ appears because of the odd harmonic of the earth's potential, 0.04409 is the so-called proper eccentricity, and other terms are due to the solar radiation pressure. The time t is measured from 37171.0 (MJD) in days. The proper eccentricity and the phase angle 91°.99 are determined from observations by the method of least squares. The acceleration is also determined to be $(4.470 \pm 0.024) \times 10^{-3}$.

Although the satellite entered the shadow after 37171 (MJD), the above expressions can follow the actual variations quite well.

References

KOZAI, Y.

1959. The earth's gravitational potential derived from the motion of Satellite 1958 β2. Smithsonian Astrophys. Obs., Special Report No. 22, pp. 1-6.

1961. The gravitational field of the earth derived from the motion of three artificial satellites. Astron. Journ., vol. 66, no. 1, in press.

MUSEN, P.

1960. The influence of the solar radiation pressure on the motion of an artificial satellite. Journ. Geophys. Res., vol. 65, pp. 1391-1396.

MUSEN, P., BRYANT, R., and BAILIE, A.

1960. Perturbations in perigee height of Vanguard I. Science, vol. 131, pp. 935-936.

PARKINSON, R. W., JONES, H. M., and SHAPIRO, I. I.

1960. Effects of solar radiation pressure on earth satellite orbits. Science, vol. 131, pp. 920-921.

SHAPIRO, I. I., and JONES, H. M.

1960. Perturbations of the orbit of the echo balloon. Science, vol. 132, pp. 1484-1486.

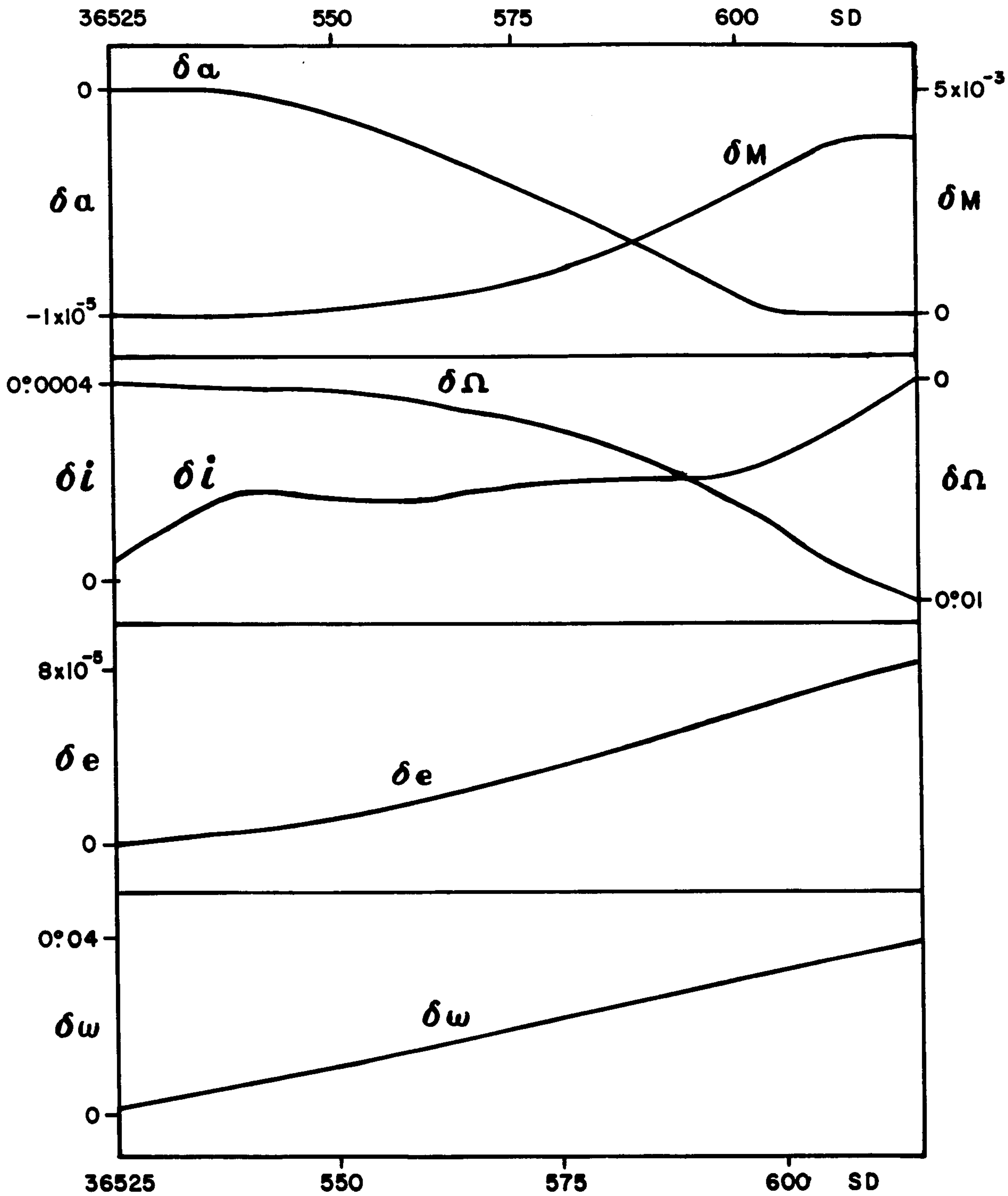

FIGURE 1.--Inequalities of the orbital elements due to the solar radiation pressure for 1958 β2.

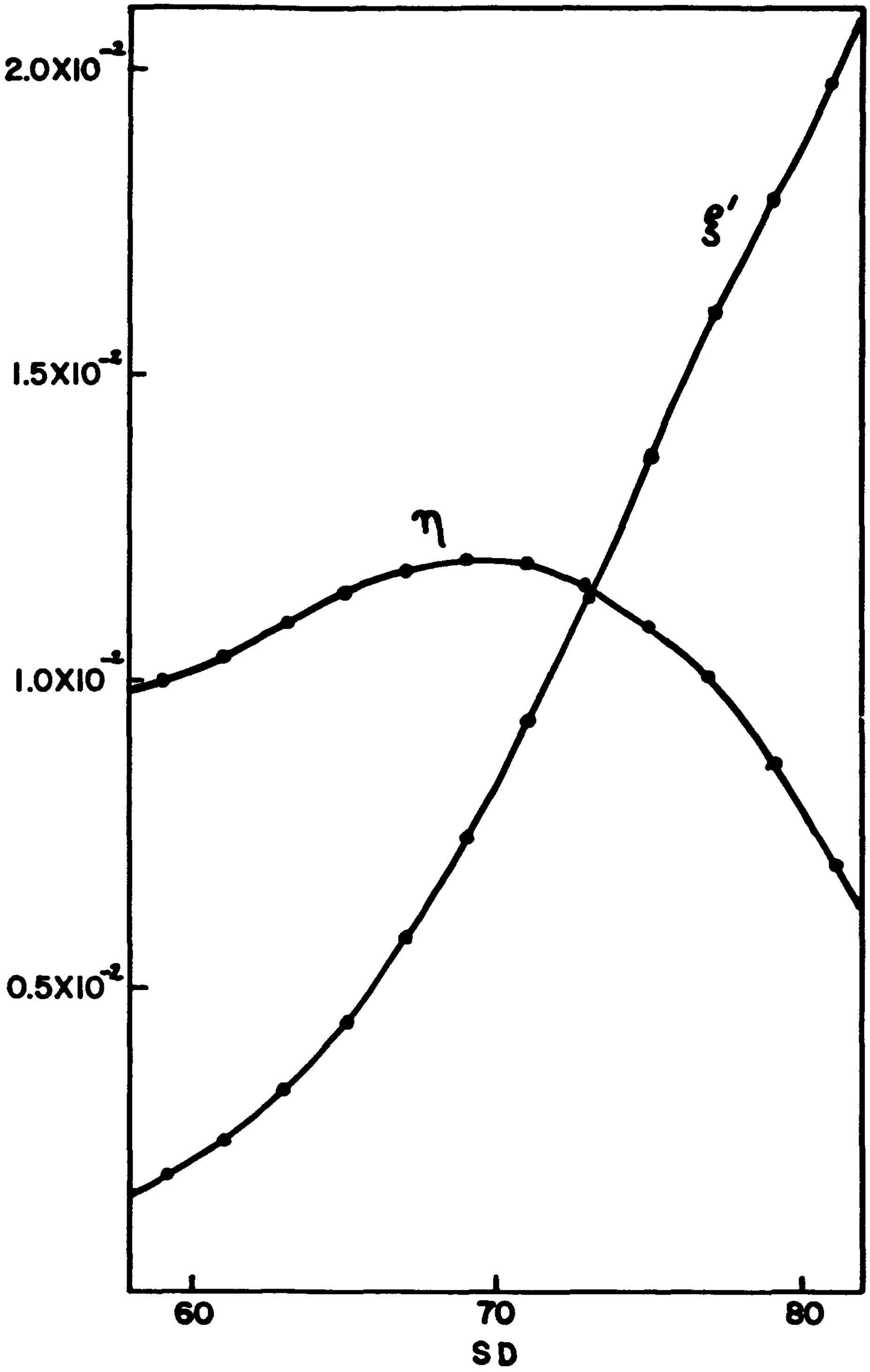

FIGURE 2. --Variations of ξ' and η for 1960 ι1 with observed values represented by dots. Epochs are given in 37100 (MJD).

Errata for SAO Special Report No. 53

The first equation on page 2 should read

$$\dot{\omega} = \frac{nJ}{a^2(1 - e^2)^2} \left(2 - \frac{5}{2} \sin^2 i\right), \quad \text{and}$$

Line 15 on page 9 should read

convenient to print $\frac{\dot{n}}{2}$ instead of $\dot{n}$ in the results of SAO mean elements.

Lines 10 through 12 on page 11 should read

given. In figures 5 and 10, the curve for $c = \frac{a_e}{297} \sin \Phi$ gives the correction needed to derive true altitude $z = q - a_e + c$, of the perigee over the international ellipsoid, where a_e is earth's equatorial radius and

Column headings on pages 13 through 19 and pages 25 through 31 should read

T (MJD)	ω	Ω	i	e	M	n	n'/2	q	N	D	σ

NOTICE

This series of Special Reports was instituted under the supervision of Dr. F. L. Whipple, Director of the Astrophysical Observatory of the Smithsonian Institution, shortly after the launching of the first artificial earth satellite on October 4, 1957. Contributions come from the Staff of the Observatory. First issued to ensure the immediate dissemination of data for satellite tracking, the Reports have continued to provide a rapid distribution of catalogues of satellite observations, orbital information, and preliminary results of data analyses prior to formal publication in the appropriate journals.

Edited and produced under the supervision of Mrs. L. G. Boyd and Mr. E. N. Hayes, the Reports are indexed by the Science and Technology Division of the Library of Congress, and are regularly distributed to all institutions participating in the U.S. space research program and to individual scientists who request them from the Administrative Officer, Technical Information, Smithsonian Astrophysical Observatory, Cambridge 38, Massachusetts.

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